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Approximation of Vector-Valued Random Variables by Constants

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Let E be a Banach space. If the closed balls in E are a compact system, then for every E-valued strongly μ-measurable random variable X, and every nondecreasing Φ: $[0, \infty) \to [0, \infty)$, there exists an $x \in E$ minimizing $\int \Phi(\|X - x\|)\, d\mu$. If $\Phi(x) = x^p$, $1 < p < \infty$, and E is strictly convex, then the operator T_p, assigning to each X the best approximating $x \in E$, is linear, if and only if the underlying probability space consists of at most 2 atoms, or $p = 2$ and E is a Hilbert space.

INTRODUCTION

Throughout this paper $(\Omega, \mathscr{A}, \mu)$ denotes a probability space. $(E, \|\ \|)$ is a Banach space; for $x \in E$ and $r \geqslant 0$, $B(x, r)$ is the closed ball centered at x with radius r. For $1 \leqslant p < \infty$, $L_p(\mu, E)$ denotes the space of equivalence classes of strongly μ-measurable E-valued functions with $\int \|X\|^p\, d\mu < \infty$. In the first section, Φ is always a nondecreasing continuous function with $\Phi(0) = 0$, Φ: $[0, \infty) \to [0, \infty)$. A sufficient condition for the existence of solutions of the following approximation problem will be given: If $X: \Omega \to E$ is a strongly μ-measurable function, find $x \in E$ such that $\int \Phi(\|X - x\|)\, d\mu = \inf\{\int \Phi(\|X - y\|)\, d\mu : y \in E\}$. For convex Φ this is a special case of a more general approximation problem considered in [1] and [2]. The results for the special case in this paper are valid for a larger class of Banach spaces E, including L_1-spaces, and the loss function Φ is more general. In the second section we restrict ourselves to strictly convex Banach spaces and $\Phi(x) = |x|^p$, $1 < p < \infty$. Except for rather trivial probability spaces the operator $T_p: L_p(\mu, E) \to E$, assigning to each $X \in L_p(\mu, E)$ the best approximating constant, is linear, if and only if $p = 2$ and E is a Hilbert space. For $E = \mathbb{R}$ the linearity of projection operators with respect to $\|\ \|_p$ has been

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investigated in [4, 6]. The additional result in this paper is: Linearity of T_2 implies that E is a Hilbert space. Finally, the relation between the Bochner integral and the approximation by a constant is discussed.

1. Existence of Best Approximants

The following facts about compact systems of sets will be needed. A system $\mathscr{C}$ of subsets of a set M is called compact, if $\mathscr{C}$ has the finite intersection property, i.e., if $\mathscr{C}_0 \subset \mathscr{C}$ and $\bigcap \mathscr{C}_1 \neq \varnothing$ for every finite subsystem of $\mathscr{C}_0$, then $\bigcap \mathscr{C}_0 \neq \varnothing$. The following remark is an easy consequence of a theorem of Alexander [3, Theorem 5.6].

Remark 1.1. If $\mathscr{C}$ is a compact system, then the system $\tau(\mathscr{C})$ of arbitrary intersections of finite unions of elements of $\mathscr{C}$ is the system of closed sets of a topology on M. Endowed with this topology, M is a quasi-compact space, i.e., $\tau(\mathscr{C})$ is a compact system.

A Banach space is said to have the intersection property (*IP*), if $\{B(x,r): x \in E, r > 0\}$ is a compact system. The following theorem shows that (*IP*) is a sufficient condition for the existence of best approximants.

THEOREM 1.2. *Let $(\Omega, \mathscr{A}, \mu)$ be a probability space, E a Banach space with (IP), $\Phi: [0, \infty) \to [0, \infty)$ a nondecreasing continuous function with $\Phi(0) = 0$. For every strongly μ-measurable function $X: \Omega \to E$, there exists $x \in E$ with $\int \Phi(\|X - x\|)\, d\mu = \inf\{\int \Phi(\|X - y\|)\, d\mu : y \in E\}$.*

Proof. Let $e = \lim_{t \to \infty} \Phi(t)$. W.l.g. we assume that $d := \inf\{\int \Phi(\|X - y\|)\, d\mu : y \in E\} < e$. Choose $\varepsilon > 0$ such that $(1 - \varepsilon)(e - \varepsilon) > d$, if $e < \infty$, $(1 - \varepsilon)\, \varepsilon^{-1} > d$, if $e = \infty$. Choose $K > 0$ such that $\mu\{\|X\| \leqslant K\} \geqslant 1 - \varepsilon$, and $M > K$, such that $\Phi(M - K) \geqslant e - \varepsilon$, if $e < \infty$, $\Phi(M - K) \geqslant \varepsilon^{-1}$, if $e = \infty$. For $y \in E$ with $\|y\| > M$ holds $\int \Phi(\|X - y\|)\, d\mu \geqslant (1 - \varepsilon)\, \Phi(M - K) > d$. Therefore $\inf\{\int \Phi(\|X - y\|)\, d\mu : y \in B(0, M)\} = d$. Let τ be the coarsest topology on $B(0, M)$ with all $B(x, r) \cap B(0, M)$, $x \in E$, $r > 0$, as closed sets. Since (*IP*) is fulfilled and, according to 1.1, $(B(0, M), \tau)$ is quasi-compact. We will show now that the function $G: B(0, M) \to [0, \infty]$ with $G(x) = \int \Phi(\|X - x\|)\, d\mu$ is lower semicontinuous (l.s.c.). Put $G_L(x) = \int \Phi(\|X - x\|)\, 1_{\{\|X\| \leqslant L\}}\, d\mu$ for $L > 0$. Then $G(x) = \sup\{G_L(x): L > 0\}$, and it suffices to prove that $G_L(x)$ is l.s.c. for $L > 0$ fixed. The sets $B(x, r) \cap B(0, M)$, $x \in E$, $r > 0$ are closed under τ, whence the function $x \to \|x + y\|$ is l.s.c. for every $y \in E$. If $\alpha_i > 0$ and $x_i \in E$, $i \in \mathbb{N}$, are given, then $x \to \sum_{i \in \mathbb{N}} \alpha_i \Phi(\|x_i - x\|)$ also is l.s.c.. Thus for every countably valued random variable Y the function $G_{L,Y}(x) = \int \Phi(\|Y - x\|)\, 1_{\{\|X\| \leqslant L\}}\, d\mu$ is l.s.c.. This proves the theorem for countably-valued X. For general X we have to show $\{x \in B(0, M): G_L(x) > a\}$ is τ-open for $a \in \mathbb{R}$, so assume $G_L(x_0) = a + \varepsilon$ for some $x_0 \in B(0, M)$, $\varepsilon > 0$. Since Φ is uniformly continuous on compact intervals, there exists $\delta > 0$ such that $0 \leqslant t \leqslant L + M$, $|t - s| \leqslant \delta$ implies $|\Phi(t) - \Phi(s)| < \varepsilon/2$. There exists a countably valued random variable $Y: \Omega \to E$ with $\|X - Y\| \leqslant \delta$ μ-a.e.. Then follows $|\Phi(\|X - x\|) - \Phi(\|Y - x\|)| < \varepsilon/2$ on the set $\{\omega \in \Omega: \|X(\omega)\| \leqslant L\}$ for all $x \in B(0, M)$. Thus, $|G_{L,Y}(x) - G_L(x)| < \varepsilon/2$ for all $x \in B(0, M)$. Hence: $\{x \in B(0, M): G_L(x) > a\} \supset \{x \in B(0, M): G_{L,Y}(x) > a + \varepsilon/2\}$, and this last set is open and contains x_0.

It follows from the above proof that the condition of 1.2 can be weakened for the case of a countably- or finite-valued random variable X.

Remark 1.3. Let $I = \mathbb{N}$, or $I = \{1, ..., n\}$ for some $n \geqslant 2$. Let $X = \sum_{i \in I} x_i 1_{A_i}$ with $x_i \in E$, $A_i \in \mathscr{A}$. If $\mathscr{C}((x_i)_{i \in I}) = \{B(x_i, r): i \in I, r > 0\}$ is a compact system, then there exists $x \in E$ with $\int \Phi(\|X - x\|)\, d\mu = \inf\{\int \Phi(\|X - y\|)\, d\mu: y \in E\}$.

In the following remarks the assertions of 1.2 and 1.3 are discussed in some special situations. The case $n = 2$ in 1.3 is simple.

Remark 1.4. For $x_1, x_2 \in E$ $\mathscr{C}(x_1, x_2)$ is always a compact system. Hence, for $\alpha_i > 0$ and every Φ there exists $x \in E$ such that $\alpha_1 \Phi(\|x_1 - x\|) + \alpha_2 \Phi(\|x_2 - x\|) = \inf\{\alpha_1 \Phi(\|x_1 - y\|) + \alpha_2 \Phi(\|x_2 - y\|): y \in E\}$. Moreover, it is easy to verify that x can be taken as $\lambda x_1 + (1 - \lambda) x_2$ with some $\lambda \in [0, 1]$.

The following example shows that the situation is more complicated if $n \geqslant 3$.

EXAMPLE 1.5. Consider the space $\mathbb{R}^3$ endowed with the norm $\|(a_i)\| = \sum_{i=1}^3 |a_i|$. Set $u_1 = (0, 0, 0)$, $u_2 = (1, -1, 0)$, $u_3 = (1, 0, -1)$. Then elementary calculus implies that $u = (1, 0, 0)$ is the unique solution of $\sum_{i=1}^3 \|u_i - u\|^2 = \inf\{\sum_{i=1}^3 \|u_i - v\|^2: v \in \mathbb{R}^3\}$. Let $H = \{(a_i) \in \mathbb{R}^3: \sum_{i=1}^3 a_i = 0\}$. It is obvious that there exists $u' \in H$ with $\sum_{i=1}^3 \|u_i - u'\|^2 = \inf\{\sum_{i=1}^3 \|u_i - v\|^2: v \in H\}$. Since $u \notin H$, holds $d := \sum_{i=1}^3 \|u_i - u'\|^2 > 3 = \sum_{i=1}^3 \|u_i - u\|^2$. Choose $\varepsilon > 0$ with $3(1 + \varepsilon)^2 < d$. After these preliminary remarks, we define a Banach space E as follows: Let $c_1 = c_2 = c_3 = 1$, and $(c_k)_{k \geqslant 4}$ be a bounded strictly increasing sequence with $c_4 = \varepsilon^{-1}$. Define $E = \{(a_i) \in l_1: \sum_{i \in \mathbb{N}} a_i c_i = 0\}$, and $\|(a_i)\| = \sum_{i \in \mathbb{N}} |a_i|$. The boundedness of (c_i) implies that E is a closed subspace of l_1. Let e_j, $j \in \mathbb{N}$, be the standard basis of l_1. Define $x_i \in E$, $i = 1, 2, 3$ by $x_1 = 0$, $x_2 = e_1 - e_2$, $x_3 = e_1 - e_3$. Assume that there exists $x = (a_i) \in E$ with $\sum_{i=1}^3 \|x_i - x\|^2 = \inf\{\sum_{i=1}^3 \|x_i - y\|^2: y \in E\}$. If $a_i = 0$ for all $i \geqslant 4$, then $\sum_{i=1}^3 \|x_i - x\|^2 \geqslant d$, but $y = e_1 - \varepsilon e_4 \in E$ fulfills $\sum_{i=1}^3 \|x_i - y\|^2 = 3(1 + \varepsilon)^2 < d$. Therefore, there exists some $j \geqslant 4$ with $a_j \neq 0$. Then $z = x - a_j e_j + a_j c_j c_{j+1}^{-1} e_{j+1} \in E$ and $\sum_{i=1}^3 \|x_i - z\|^2 < \sum_{i=1}^3 \|x_i - x\|^2$, since $(c_i)_{i \geqslant 4}$ is strictly increasing. Hence there exists no $x \in E$ with $\sum_{i=1}^3 \|x_i - x\|^2 = \inf\{\sum_{i=1}^3 \|x_i - y\|^2: y \in E\}$. According to 1.3, $\mathscr{C}(x_1, x_2, x_3)$ is not a compact system in this example.

The assumption "$\mathscr{C}(x_1,..., x_n)$ compact for every $n \geqslant 2$ and all $x_1,..., x_n \in E$" is not strong enough to imply the existence of best approximants for countably valued random variables.

EXAMPLE 1.6. If $E = c_0$, the system $\mathscr{C}(x_1,..., x_n)$ is always a compact system, but the system of all closed balls in c_0 is not compact. Take $\Omega = \mathbb{N} \cup -\mathbb{N}$ and define a probability measure $\mu \mid \mathscr{P}(\Omega)$ by $\mu(\pm n) = 2^{-n-1}$. If $X: \Omega \to c_0$ is defined by $X(\pm n) = \pm e_1 + 2e_n$, $n \in \mathbb{N}$, and Φ is a strictly convex function, then there exists no $x \in c_0$ with $\int \Phi(\|X - x\|)\, d\mu = \inf\{\int \Phi(\|X - y\|)\, d\mu : y \in c_0\}$.

The following remark shows that there are many spaces, which fulfill (IP).

Remark 1.7. (i) Every dual space has (IP).

(ii) Every weak-$*$-closed subspace of a dual space has (IP).

(iii) If there exists a linear projection $\pi: E^{**} \to E$ with $\|\pi\| \leqslant 1$, then E has (IP).

(iv) For every σ-finite measure space $(\Sigma, \mathscr{F}, \nu)$ $L_1(\nu, \mathbb{R})$ has (IP).

Proof. (i) and (ii) follow from the weak-$*$-compactness of closed balls in dual spaces.

(iii) If $\mathscr{C} = \{B(x_i, r_i)\}$ is a system of closed balls in E, then define $B'(x_i, r_i) = \{y \in E^{**} : \|x_i - y\| \leqslant r_i\}$. If every finite subsystem of $\mathscr{C}$ has nonvoid intersection, then by (i) there exists $x' \in \bigcap B'(x_i, r_i)$. Then $x = \pi(x') \in \bigcap B(x_i, r_i)$.

(iv) For $E = L_1(\nu, \mathbb{R})$ E^{**} can be identified with the space of bounded additive set functions on $\mathscr{F}$, which vanish on μ-null sets [7, Th. 2.3]. For every $\rho \in E^{**}$ let $\pi(\rho)$ be a ν-density of the σ-additive part of the Yoshida–Hewitt decomposition of ρ [7, Th. 1.23]. Then $\pi: E^{**} \to E$ is a linear projection with $\|\pi\| \leqslant 1$. Hence (iii) implies (iv).

2. Linearity of Best Approximation with Respect to $\| \; \|_p$

In this section we will assume that E is a strictly convex Banach space, and that (IP) is fulfilled. Then, for every probability space $(\Omega, \mathscr{A}, \mu)$ and every $p \in (1, \infty)$ we define an operator $T_p: L_p(\mu, E) \to E$ by

$$\int \|X - T_p X\|^p \, d\mu = \inf \left\{ \int \|X - y\|^p \, d\mu : y \in E \right\}. \tag{2.1}$$

Since E is strictly convex and Φ is a strictly convex function for every $p \in (1, \infty)$, T_pX is uniquely determined by (2.1).

THEOREM 2.2. *T_p is linear if and only if*

(a) *Ω is the union of 2 μ-atoms, or*

(b) *$p = 2$ and E is a Hilbert space.*

Proof. 1. Let $A_i \in \mathscr{A}$, $i = 1, 2$, be disjoint sets with $\Omega = A_1 \cup A_2$, $\alpha_i = P(A_i)$, and $x_i \in E$ for $i = 1, 2$. Using 1.4 and differential calculus, we obtain

$$T_p(x_1 1_{A_1} + x_2 1_{A_2}) = \beta_1 x_1 + \beta_2 x_2$$
$$\text{with} \quad \beta_i = \alpha_i^r/(\alpha_1^r + \alpha_2^r) \qquad i = 1, 2, \quad r = (p-1)^{-1}. \tag{2.3}$$

(2.3) clearly implies that T_p is linear, if (a) is fulfilled.

2. If (b) is fulfilled, then T_p is linear, since $T_p = T_2$ is a projection on a closed subspace of the Hilbert space $L_2(\mu, E)$.

3. Assume that (a) is not fulfilled and $p > 2$. There exist disjoint sets $A_i \in \mathscr{A}$, $i = 1, 2, 3$, with $\alpha_i = \mu(A_i) > 0$ and $\Omega = A_1 \cup A_2 \cup A_3$. Put $r = (p-1)^{-1}$. W.l.g. we assume $\alpha_3^r + (1-\alpha_3)^r \geqslant \alpha_i^r + (1-\alpha_i)^r$ for $i = 1, 2$. Take an arbitrary $x \in E - \{0\}$ and define $X_i = x 1_{A_i}$ for $i = 1, 2$. According to (2.3) we have $T_p X_i = \beta_i x$, $i = 1, 2$, with $\beta_i = \alpha_i^r/(\alpha_i^r + (1-\alpha_i)^r)$, and $T_p(X_1 + X_2) = \beta x$ with $\beta = (\alpha_1 + \alpha_2)^r/(\alpha_3^r + (1-\alpha_3)^r)$. $0 < r < 1$ implies $\alpha_1^r + \alpha_2^r > (\alpha_1 + \alpha_2)^r$, and therefore $\beta_1 + \beta_2 \geqslant (\alpha_1^r + \alpha_2^r)/(\alpha_3^r + (1-\alpha_3)^r) > \beta$. This inequality shows that T_p is not linear. The case $1 < p < 2$ runs similarly.

4. Assume that (a) is not fulfilled, $p = 2$, and T_p is linear. Let $A_i \in \mathscr{A}$, $i = 1, 2, 3$, be as in 3. For $i = 1, 2, 3$ put $\lambda_i = \alpha_i/(\alpha_1 + \alpha_2)$. We will show

$$\lambda_1 \|x_1\|^2 + \lambda_2 \|x_2\|^2 \leqslant \lambda_1 \lambda_2 \|x_1 - x_2\|^2 + \|\lambda_1 x_1 + \lambda_2 x_2\|^2$$
$$\text{for all } x_1, x_2 \in E. \tag{2.4}$$

Define: $c_0 = \inf\{c > 0\colon$ For all $x_1, x_2 \in E$ holds $\lambda_1 \|x_1\|^2 + \lambda_2 \|x_2\|^2 \leqslant \lambda_1 \lambda_2 \|x_1 - x_2\|^2 + c \|\lambda_1 x_1 + \lambda_2 x_2\|^2\}$. Let $x_1, x_2 \in E$ be given. Take $x_3 = -\lambda_3^{-1}(\lambda_1 x_1 + \lambda_2 x_2)$ and $X = \sum_{i=1}^3 x_i 1_{A_i}$. Then (2.3) and the linearity of T_2 imply $T_2 X = 0$. For $\alpha \in (0, 1]$ put $y_\alpha = \alpha(\lambda_1 x_1 + \lambda_2 x_2)$. According to (2.1), we have $\sum_{i=1}^3 \lambda_i \|x_i\|^2 \leqslant \sum_{i=1}^3 \lambda_i \|x_i - y_\alpha\|^2$. This is equivalent to:

$$\lambda_1 \|x_1\|^2 + \lambda_2 \|x_2\|^2 \leqslant \lambda_1 \|x_1 - y_\alpha\|^2 + \lambda_2 \|x_2 - y_\alpha\|^2$$
$$+ (2\alpha + \alpha^2 \lambda_3) \|\lambda_1 x_1 + \lambda_2 x_2\|^2. \tag{$*$}$$

Putting $\alpha = 1$ in ($*$), we obtain $\lambda_1 \|x_1\|^2 + \lambda_2 \|x_2\|^2 \leqslant \lambda_1 \lambda_2 \|x_1 - x_2\|^2 +$

$(2+\lambda_3)\,\|\lambda_1 x_1+\lambda_2 x_2\|^2$. Hence, $c_0 \leqslant 2+\lambda_3<\infty$. Starting from $(*)$ and using the definition of c_0, we can write

$$\begin{aligned}
&\lambda_1\|x_1\|^2+\lambda_2\|x_2\|^2\\
&\quad\leqslant \lambda_1\lambda_2\,\|(x_1-y_\alpha)-(x_2-y_\alpha)\|^2\\
&\qquad + c_0\,\|\lambda_1(x_1-y_\alpha)+\lambda_2(x_2-y_\alpha)\|^2+(2\alpha+\alpha^2\lambda_3)\,\|\lambda_1 x_1+\lambda_2 x_2\|^2\\
&\quad= \lambda_1\lambda_2\,\|x_1-x_2\|^2+(c_0(1-\alpha)^2+2\alpha+\alpha^2\lambda_3)\,\|\lambda_1 x_1+\lambda_2 x_2\|^2.
\end{aligned}$$

Whence we have for every $\alpha\in(0,1]$:

$$c_0\leqslant c_0(1-\alpha)^2+2\alpha+\alpha^2\lambda_3.$$

This implies $c_0\leqslant 1$, which proves (2.4). The fact that E is a Hilbert space, if (2.4) is fulfilled, is stated in the following proposition.

PROPOSITION 2.5. *If* (2.4) *is fulfilled for some* $\lambda_i>0$ *with* $\lambda_1+\lambda_2=1$, *then* E *is a Hilbert space.*

Proof. We show first

$$\|x_1\|^2+\|x_2\|^2=\|\gamma_1 x_1+\gamma_2 x_2\|^2+\|\gamma_2 x_1-\gamma_1 x_2\|^2$$
$$\text{for all } x_1, x_2\in E;\ \text{with } \gamma_i=\lambda_i^{1/2} \text{ for } i=1,2. \qquad (2.5)$$

If $x_1, x_2\in E$ are given, then using (2.4), we obtain

$$\begin{aligned}
\|x_1\|^2+\|x_2\|^2&=\lambda_1\,\|\gamma_1^{-1}x_1\|^2+\lambda_2\,\|\gamma_2^{-1}x_2\|^2\\
&\leqslant \lambda_1\lambda_2\,\|\gamma_1^{-1}x_1-\gamma_2^{-1}x_2\|^2+\|\lambda_1\gamma_1^{-1}x_1+\lambda_2\gamma_2^{-1}x_2\|^2\\
&=\|\gamma_1 x_1+\gamma_2 x_2\|^2+\|\gamma_2 x_1-\gamma_1 x_2\|^2.
\end{aligned}$$

An application of this inequality to $y_1=\gamma_1 x_1+\gamma_2 x_2$ and $y_2=\gamma_2 x_1-\gamma_1 x_2$ instead of x_1 and x_2, yields the converse inequality.

Next we prove

$$\|x-y\|=\|x+y\|\Rightarrow\|x-\gamma y\|=\|x+\gamma y\| \qquad \text{for all } x, y\in E;$$
$$\text{with}\quad \gamma=(1+\gamma_1)/(1-\gamma_1). \qquad (2.6)$$

If $x, y\in E$ are given and $\|x-y\|=\|x+y\|$, then we apply (2.5) with $x_1=x-y$ and $x_2=(1-\gamma_1)\,\gamma_2^{-1}x+(1+\gamma_1)\,\gamma_2^{-1}y$. After a short computation we obtain $\|x-\gamma y\|=\|x+\gamma y\|$. Since $\gamma>0$, $\gamma\neq 1$, (2.6) implies that E is a Hilbert space, according to [5] (I_1).

The following corollary is an immediate consequence of Theorem 2.2 and (2.3).

COROLLARY 2.7. *$T_p X = \int X\, d\mu$ for every $X \in L_p(\mu, E)$, if and only if*

(a) *Ω is a μ-atom, or the union of 2 μ-atoms and $p = 2$, or*

(b) *$p = 2$ and E is a Hilbert space.*

In the general case the relation between the Bochner integral and best approximation by a constant can be stated as follows.

Remark 2.8. If E is a strictly convex Banach space, then the Bochner integral is the unique linear continuous operator $T: L_1(\mu, E) \to E$, which fulfills

$$\int \|X - TX\|^2\, d\mu = \inf \left\{ \int \|X - y\|^2\, d\mu : y \in E \right\}$$

for every μ-measurable $X: \Omega \to E$, which attains only 2 values.

Proof. From (2.3) and the linearity of T follows $TX = \int X\, d\mu$ for every finitely valued μ-measurable function $X: \Omega \to E$. Then $TX = \int X\, d\mu$ for every $X \in L_1(\mu, E)$, since T is continuous.

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